have been incorporated from "The Teaching and History of Mathematics in the U. S., by Professor Florian Cajori (1890), to which has been added such matter as was necessary to bring the sketch down to date.]

SOME GENERAL FORMULAS FOR SQUARE NUMBERS WITH APPLICATIONS.

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Let us first examine the Formula, $(m^2+n^2)^2=(m^2-n^2)^2+(2mn)^2$.....(1).

This is identically true for all values of m and n and expresses one square number as the sum of two other square numbers. Give m and n different values and find as follows:

m	n		m	n	
2	1	$5^2=3^2+4^2$	5	1	$26^2=24^2+10^2$
2	1	$10^2=8^2+6^2$	5	2	$29^2=21^2+20^2$
3	2	$13^2=5^2+12^2$	5	3	$34^2=16^2+30^2$
4	1	$17^2=15^2+8^2$	5	4	$41^2=9^2+40^2$
4	2	$20^2=12^2+16^2$			
4	3	$25^2=7^2+24^2$			

In reference to this formula. we remark that in order to produce all possible sets of numbers answering to the conditions, the numbers of no set being equimultiples of those of any other set, then evidently m and n must be prime numbers and cannot both be even.

Moreover m and n cannot both be odd, for in that case m^2+n^2, m^2-n^2, and $2mn$ would all be even, and of the form $2(m^2_1+n^2_1)$, $2(m^2_1-n^2_1)$ and $2(2m_1n_1)$, and would give numbers twice as large as the values m and n would give. This is shown above; for $m=3$, $n=1$; $m=5$, $n=1$; etc.

Hence, m and n are both prime, one is odd the other even.

Hence, m^2+n^2 and m^2-n^2 are both odd and $2mn$ is even.

For all sets of values, we have the following *General* formula:

$[p(m^2+n^2)]^2=[p(m^2-n^2)]^2+(2pmn)^2$....(2), in which m and n are prime numbers, one of which is odd and the other even, and p any number. Let $m=2$ and $n=1$; then

$p=2$ gives $10^2=6^2+8^2$ $\quad$ $p=3$ gives $15^2=9^2+12^2$

$p=4$ gives $20^2=12^2+16^2$ $\quad$ $p=5$ gives $25^2=15^2+20^2$, etc.

None of these sets could be given by (1) m and n being prime, and one odd, the other even; and all sets given by (2), p being any number not 2 or a square, could not be given by (1) at all.

II. To find three square numbers, whose sum is a square number.

Making $m^2=a^2_1+a^2_2$ and $n^2=a^2_3$ in eq. (1) gives,

$$(a^2_1+a^2_2+a^2_3)^2=(a^2_1+a^2_2-a^2_3)+(2a_1a_3)^2+(2a_2a_3)^2 \ldots\ldots(3).$$

The same may be at once written by analogy. This furnishes a solution to the problem:

To find a parallelopipedon whose diagonal and edges are integral numbers.

If in (3) we make $a_1=a_2$ we have a solution to the problem: To find a parallelopipedon whose base is a square, and diagonal and edges are integral numbers.

Ex. Let $a_1=a_2=2$ and $a_3=1$. Then $9^2=7^2+4^2+4^2$.

The sides of the base are 4, altitude 7, and diagonal 9. If in (3) we assign any values to a_1 and a_2, we have a solution to the problem: To find a parallelopipedon whose base is any rectangle, and diagonal and edges integral numbers.

Ex. Let $a_1=2$, $a_2=3$, and $a_3=1$. Then $14^2=12^2+4^2+6^2$ or $7^2=6^2+3^2+2^2$. In the latter, the sides of the base are 2 and 3, altitude 6 and diagonal 7.

III. To find n square numbers, whose sum is a square number.

Make $m^2=a^2_1+a^2_2+\ldots\ldots+a^2_{n-1}$, and $n^2=a^2_n$ in eq. (1) and find directly,

$$(a^2_1+a^2_2+\ldots\ldots a^2_n)^2=(a^2_1+a^2_2+\ldots\ldots+a^2_{n-1}-a^2_n)^2$$
$$+(2a_1a_n)^2+(2a_2a_n)^2+\ldots\ldots(2a_{n-1}a_n)^2\ldots\ldots(4).$$

We may assign values to $a_1, a_2, \ldots\ldots a_{n-1}$ in (4) so as to satisfy any possible conditions. If we make these quantities equal we will have a square number, equal to the sum of n square numbers of which $n-1$ are equal.

If $a_1=1$, $a_2=2\ldots\ldots a_{n-1}=n-1$, and $a_n=1$, we have, $[1^2+2^2+\ldots\ldots+(n-1)^2+1^2]=[1^2+2^2+\ldots\ldots+(n-1)^2-1^2]+[2^2+4^2+\ldots\ldots 2^2(n-1)^2.]$

IV. To find a square number, which added to *any* given number will make a square number.

Let S=the given number, y^2 the square number, to be added to make a square number.

Put $S+y^2=x^2$. $\therefore (x+y)(x-y)=S=a\times b$.

Put $x+y=a$ and $x-y=b$, then $S=ab$, $x=\dfrac{a+b}{2}$, and $y=\dfrac{a-b}{2}$.

Since x and y are integral a and b must be both odd, or both even.

$$\therefore ab+\left(\frac{a-b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2 \text{ or, } ab+\left(\frac{a-b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2;$$

$$\text{or, } ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2.$$

Hence, any number which is the product of two even or of two odd factors can be expressed as the difference between two squares. Since S is entirely unresticted, this equation is entirely general, and very prolific in results. For example, S may represent any series, complete or incomplete, or

the sums or differences of several series complete or incomplete. We will give several examples, but an indefinite number could be adduced.

a. Take $S=1+3+5+\ldots.$ to n terms, $=n^2$.

1. Let $n=4$, then $S=4^2=8\times2=a\times b$. $\frac{a-b}{2}=3$ and $\frac{a+b}{2}=5$ and we have, $1+3+5+7+3^2=5^2$, or $4^2+3^2=5^2$.

2. Let $n=5$, then $S=5^2=25\times1=a\times b$ and we have, $1+3+5+7+9+12^2=13^2$; or $5^2+12^2=13^2$.

3. Let $n=1000$, then $S=(1000)^2=2500\times400$ and we have $(1000_2)+(1050)^2=(1450)^2$.

b. Take $S=1^2+2^2+\ \ldots n^2$.

1. Let $n=5$, then $S=55=55\times1$ or 11×5.

Let $a=55$ and $b=1$, and we have, $1+2^2+\ldots.+n^2+27^2=28^2$.

Or let $a=11$ and $b=5$, and we have, $1^2+2^n+\ldots\ldots+n^2+3^2=8^2$.

These are the only *two* solutions of this case.

2. Let $S=1^2+2^2+\ldots.+10^2=385\times1=77\times5=55\times7=35\times11$. Taking $a=35$ and $b=11$, we have $1+2+\ldots.+10^2+12^2=23^2$. There are three other solutions.

c. Take $S=(1^2+2^2+\ldots.n^2)-(1+2^2+\ldots.m^2)-p^2-q\pm\ldots.$

1. Let $n=7$, $m=2$, $v^2=5^2$, $q=11$.

Then, $S=3^2+4^2+6^2+7^2-11=99=11\times9$. $\therefore\ 3^2+4^2+6^2+7^2-11+1^2=10^2$.

When $S=$any $(n-1)$ square numbers, this formula furnishes another solution to the problem: To find n square numbers whose sum is a square number.

1. Find eleven square numbers whose sum is a square.

Take $S=1^2+2^2+\ldots.10^2=385=35\times11$.

$\therefore\ 1^2+2^2+\ldots.10^2+12^2=23^2$.

Find fourteen square numbers, whose sum is a square number.

Take $S=1^2+2^2+\ldots.13^2=819\times1=273\times3=91\times9=63\times13=39\times21$.

Using 63 and 13, $1^2+2^2+\ldots.13^2+25^2=33^2$.

Using 91 and 9, $1^2+2^2+\ldots.13^2+41^2=50^2$ etc., etc.

In case S, though not a prime number, cannot be separated into two factors, both odd or both even, S must be an even number, in which case subtract any odd square number, say $(2n_1-1)^2$, then factor the remainder and proceed as above.

Example. To find two square numbers, one of which being added to, and the other taken from $S=1^2+2^2+\ldots.16^2=1496$ will make a square number.

$$1496=15^2+1271;\ 1271=41\times 31=36^2-5^2.$$
$$\therefore\ S-15^2=36^2-5^2,\ \text{or}\ S-15^2+5^2=36^2,$$
$$\text{or}\ 1^2+2^2+\ldots.14^2+16^2+5^2=36^2.$$

COROLLARY.—If the square number $(2n^2_1-1)^2$ taken from S is one of the square numbers in S, as in the above example and which may always be the case, the formula still furnishes the means of finding n square numbers whose sum is a square number.

We may write at once, $(n+1)^2-1^2=n(n+2)$

$$(n+2)^2-2^2=n(n+4)$$
$$(n+m)^2-m^2=n(n+2m).$$

Hence we see the form that two factors must have, in order that their product may be equal to the difference between two squares.

We see that the factors are both odd or both even.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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CHAPTER THIRD.

On the Continuity of Space.

[Continued from the July Number].

It is here thought best to interpolate some expository matter regarding parts of elementary geometry which involve the difficult idea of *continuity.*

When a mathematician says: "There may be a triangle whose angle-sum differes from a straight angle by less than any *given* finite angle however small," the meaning is simply, "give me geometrically any one particular finite angle you choose, and I will prove geometrically that a triangle may exist the sum of whose three interior angles differs from two right angles by less than that particular angle you have given me."

The problem: "To construct a triangle whose angle-sum differs from a straight angle by less than any given finite angle however small," means, "if any one particular finite angle is given graphically, show how geometrically to construct a triangle whose angle-sum differs from two right angles by less than that one particular *given* finite angle."